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Pricing with a Smile

Bruno Dupire

Bloomberg

The Black–Scholes model (see Black and Scholes, 1973) gives options prices as a function of volatility. If an option price is given by the market we can invert this relationship to get the *implied* volatility.

If the model were perfect, this implied value would be the same for all option market prices, but reality shows this is not the case. Implied Black–Scholes volatilities strongly depend on the maturity and the strike of the European option under scrutiny. If the implied volatilities of at-the-money (ATM) options on the Nikkei 225 index are 20% for a maturity of six months and 18% for a maturity of one year, we are in the uncomfortable position of assuming that the Nikkei oscillates with a constant volatility of 20% for six months but also oscillates with a constant volatility of 18% for one year.

It is easy to solve this paradox by allowing volatility to be time-dependent, as Merton did (see Merton, 1973). The Nikkei would first exhibit an instantaneous volatility of 20% and subsequently a lower one, computed by a forward relationship to accommodate the one-year volatility. We now have a single process, compatible with the two option prices. From the term structure of implied volatilities we can infer a time-dependent instantaneous volatility, because the former is the quadratic mean of the latter. The spot process S is then governed by the following stochastic differential equation:

$$\frac{\mathrm{d}S}{S} = r(t)\,\mathrm{d}t + \sigma(t)\,\mathrm{d}W$$

where $r(t)$ is the instantaneous forward rate of maturity t implied from the yield curve.

Some Wall Street houses incorporate this temporal information in their discretisation schemes to price American or path-dependent options.

However, the dependence of implied volatility on the strike, for a given maturity (known as the *smile* effect) is trickier. Researchers have attempted to enrich the Black–Scholes model to compute a theoretical "smile". Unfortunately, they have to introduce a non-traded source of risk such as jumps, stochastic volatility or transaction costs, thus losing the completeness (ability to hedge options with the underlying asset) of the model.[1] Completeness is of the highest value: it allows for arbitrage pricing and hedging.

Therefore, we must ask whether it is possible to build a spot process that:

- ❑ is compatible with the observed smiles at all maturities, and
- ❑ keeps the model complete.

More precisely, given the arbitrage-free prices $C(K, T)$ of European calls of all strikes K and maturities T, is it possible to find a risk-neutral process for the spot in the form of a diffusion,

$$\frac{\mathrm{d}S}{S} = r(t)\,\mathrm{d}t + \sigma(S, t)\,\mathrm{d}W$$

where the instantaneous volatility σ is a deterministic function of the spot and of the time?

This would extend the Black–Scholes model to make full use of its diffusion setting without increasing the dimension of the uncertainty. We would have the features of a one-factor model (hence easily discretisable) to explain all European option prices. We could then price and hedge any American or path-dependent options (even for European options, the knowledge of the whole process is necessary for hedging). We would also know which volatility to use to price a barrier option and how to hedge a compound option. It is quite simple to work on a discretised version of the spot, as we show later, but here we also give an analytical treatment, which is more revealing.

If the spot price follows a one-dimensional diffusion process, then the model is complete and option prices can be computed by discounting an expectation with respect to a "risk-neutral" probability under which the discounted spot has no drift (but retains the same diffusion coefficient).

More precisely, path-dependent options are priced as the discounted expected value of their terminal payoff over all possible paths. In the case of European options, this boils down to an expectation about the terminal values of the spot (which can be seen as bundling the paths that end at a same point).

It follows that knowledge of the prices of all path-dependent options is equivalent to knowledge of the full (risk-neutral) diffusion process of the spot; knowing all European option prices merely amounts to knowing the probability densities of the spot at different times, conditional on its current value.

The full diffusion contains much more information than the conditional laws, as distinct diffusions may generate identical conditional laws. For instance, a Gaussian process with mean reversion can generate the same conditional laws as another Gaussian process with volatility decreasing over time. However, as we shall see, if we restrict ourselves to risk-neutral diffusions, the ambiguity is removed and we can retrieve from the conditional laws the unique risk-neutral diffusion from which they come. This result is interesting in itself but we will also exploit its consequences in hedging terms.

A DIFFUSION FROM PRICES

We can gain considerable clarity without losing much in generality by assuming that the interest rate is zero. For a given maturity T, the collection of option prices of different strikes $C(K, T)$ – which in practice requires a smooth interpolation from a few points – yields the risk-neutral density function φ_T of the spot at time T through the relationship:

$$C(K, T) = \int_0^\infty \max(S - K, 0)\varphi_T(S)\,\mathrm{d}S$$

which we differentiate twice with respect to K to obtain:

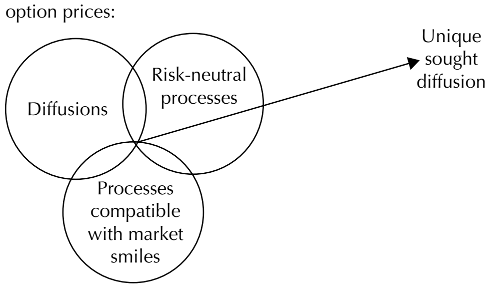
$$\varphi_T(K) = \frac{\partial^2 C}{\partial K^2}(K, T)$$

which is the risk-neutral probability density of the spot being equal to K at time T. We recall that European option prices are equivalent to the densities φ_T and that path-dependent option prices are equivalent to the diffusion process. We are then left with an interesting stochastic problem – with the notation (x, t) instead of (K, T): knowing all the densities conditional on an initial fixed (x_0, t_0), is there a unique diffusion process $\mathrm{d}x = a(x, t)\,\mathrm{d}t + b(x, t)\,\mathrm{d}W$ which generates these densities?

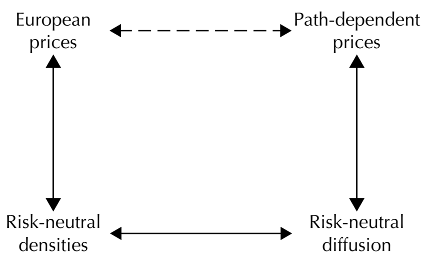
The solution in general is not unique; however, if we restrict ourselves to *risk-neutral* diffusions, we can recover, under some technical assumptions, a unique diffusion process from the φ_T (see Figure 1). The interest rate being zero, we pay attention only to martingale (ie, driftless) diffusions $\mathrm{d}x = b(x, t)\,\mathrm{d}W$.

Figure 1 A unique diffusion process

If we restrict ourselves to diffusions, there is a unique risk-neutral (drift equal to the short-term rate) process for the spot which is compatible with European option prices:



This means that, if we assume the spot is following a diffusion process, we can obtain exotic option prices from European option prices through the scheme:



Thanks to the Fokker-Planck equation, we can – after some maths (see Dupire, 1993b) – write

$$\frac{b^2(K,T)}{2}\frac{\partial^2 C}{\partial K^2} = \frac{\partial C}{\partial T} \qquad \textbf{(E)}$$

where $C(S,t,K,T)$ denotes the premium at time t for a spot S of a European call of strike K and maturity T.

Both derivatives are positive by arbitrage (butterfly for the convexity and conversion for the maturity). Equation (E) can be used to determine b, as

$$\frac{\partial^2 C}{\partial K^2} \quad \text{and} \quad \frac{\partial C}{\partial T}$$

are known from the market smiles. We can infer the instantaneous volatility at time T for a spot equal to K from the knowledge of the option prices of maturities and strikes around T and K, which is our primary purpose. Going back to the spot process $\mathrm{d}S/S = \sigma(S,t)\,\mathrm{d}w$, we indeed obtain the instantaneous volatility by

$$\sigma(S,t) = \frac{b(S,t)}{S}$$

A NEW WAY TO COMPUTE PRICE

Equation (E) can also be interpreted in another fashion. If b is known, it establishes a relationship between the price as of today of call options of varying maturities and strikes.

Equation (E) has the same flavour as, but is distinct from, the classical Black–Scholes partial differential equation which involves, for a *fixed option* (ie, K and T fixed), derivatives with respect to the current time and value of the spot. With zero interest rates, the Black–Scholes equation takes the form:

$$-\frac{b^2(S,t)}{2}\frac{\partial^2 C}{\partial S^2} = \frac{\partial C}{\partial t} \qquad \textbf{(BS)}$$

Equations (E) and (BS) can be thought as being dual to each other. However, the relationship is not so universal, as (BS) applies to any contingent claim, though (E) holds only because the intrinsic value of a call happens to be the second integral of a Dirac function. It is very fortunate that the market trades this particular payoff!

It also provides an algorithm to compute an option price through a forward tree and even the price of many different options in a single sweep of the tree!

To price the (K, T) call, we build a tree with its root at (K, T), expanding backward in time up to the current date where it is fed by an intrinsic value which is the value today of an option of immediate maturity. Pricing is performed forward in time by taking the discounted expectation at each node until the root (K, T) is reached and the premium can be collected (see Figure 2).

An internal node of the tree will be labelled with today's value of a European call where strike and maturity correspond to this node, as opposed to a standard tree where each node carries the premium of a *fixed option* at a future time and spot associated to that node.

It is indeed possible to compute b numerically from the relation (E) obtained from the continuous time and price analysis, and to discretise the associated spot process with explicit recombining binomial (see Nelson and Ramaswamy, 1990) or trinomial (see Hull and White, 1990) schemes. We prefer however to present a construction which makes use of a new technique widely used for interest rate model fitting: forward induction, (see Jamshidian, 1991; Hull and White, 1992) as it can be understood without any stochastic machinery.

It is worth stressing that it is quite easy to find a set of coefficients that price options correctly, since degrees of freedom are in superabundance compared to the constraints. The situation is analogous to the one encountered in the continuous case, where various diffusions could generate the same densities. However, imposing the martingale condition (risk-neutrality) in the discrete time setting at each node gives additional constraints. This extra structure is a key point in our pricing/hedging approach but existence and uniqueness are in general not achieved by a simplistic discretisation. As we shall see, a trinomial one does ensure existence and uniqueness of the discretised process, through a parsimonious use of its degrees of freedom (the weights carried by the connections).

We build a trinomial tree with equally spaced time-steps. The ratio of price-step over time-step, which determines the opening of the tree, has to be large enough to cater for the local variance of the

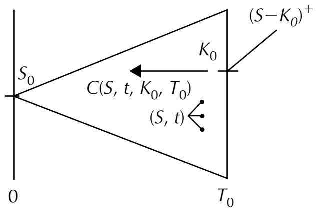
Figure 2 A new way to price options

Spot follows $dx_t = b(x_t, t)\,dW_t$ (interest rates are 0).
Two ways to compute $C(S_0, 0, K_0, T_0)$:

Black–Scholes PDE (BS)
K, T fixed

$$\frac{\partial C}{\partial t} = -\frac{b^2(S, t)}{2}\frac{\partial^2 C}{\partial S^2}$$

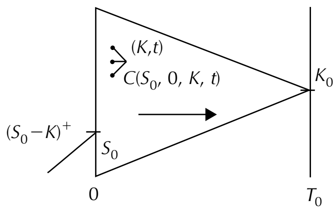
computes $C(S, t, K_0, T_0)$



Fokker-Planck (E)
S, t fixed

$$\frac{\partial C}{\partial T} = \frac{b^2(K,T)}{2}\frac{\partial^2 C}{\partial K^2}$$

computes $C(S_0, 0, K, T)$

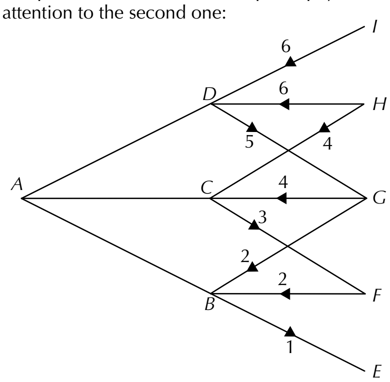


In both cases, $C(S_0, 0, K_0, T_0)$ collected at the root of the tree.

process. This condition is equivalent to the one that guarantees the stability of explicit discretisations of a partial differential equation. If the market Black–Scholes smiles are not too pronounced, equal steps on the logarithm of the spot are best. If the initial guess of the opening is not high enough, it should be increased to ensure that the procedure described below can be carried out. Weights will be assigned to the connections, which will allow us to compute the discounted probability of each path and hence to value any

Figure 3 Building the tree

We assume the connections have been computed over the first time step and pay attention to the second one:



1 is computed forward through the Arrow-Debreu prices of *B* and *E*
2 are computed backward through 1 and the zero coupon and the spot at period 2
3 is computed forward through the Arrow-Debreu prices of *B*, *C*, *F* and 2
4 are computed backward through 3 and the zero coupon and the spot at period 2
5 is computed forward through the Arrow-Debreu prices of *B*, *C*, *D*, *G*, 2 and 4
6 are computed backward through 5 and the zero coupon and the spot at period 2

Arrow-Debreu profiles of *H* and *I* need not be exploited, as they are necessarily correctly priced by the tree. In effect, they are spanned by the Arrow-Debreu profiles of *E*, *F* and *G*, the zero coupon and *S*, which are correctly priced.

path-dependent option. It is possible to reduce the complexity of the computation in many cases.

At each discrete date, all profiles consisting of continuous piecewise linear functions with break points located at inner nodes of the tree are required to be correctly priced by the tree. At the nth step, any such profile is uniquely characterised by the value it takes on the $2n + 1$ nodes of that step, thus the space of all profiles is of dimension $2n + 1$. This contains the zero coupon, the asset itself and all calls (and puts) whose strikes are the inner nodes. With each node we associate an Arrow-Debreu profile whose value is 1 on this node and 0 on the others.

A node is labelled (n, i) with n denoting the time-step and i the price-step. Its associated Arrow-Debreu price is denoted $A(n, i)$ and the weight of the connection between nodes (n, i) and $(n + 1, j)$, $j = i - 1$, i, or $i + 1$ is denoted $w(n, i, j)$. Arrow-Debreu prices are computed from market prices, as prices of portfolios of European calls, spot and cash positions. The weights are computed through the tree in a forward fashion.

We can exploit two types of relations:

- ❑ forward relations, which relate the Arrow-Debreu price of a node to the Arrow-Debreu prices of its immediate predecessors;
- ❑ standard backward relations, which link the value of a contingent claim at a node to its value at the immediate successors. We apply

this relation to two simple claims: a unit of cash and a unit of the spot, both to be received one time-step later (see Figure 3).

The generic step of the algorithm is:

Compute $w(n, i, i-1)$ from $A(n+1, i-1)$, $A(n, i)$, $A(n, i-1)$, $A(n, i-2)$, $w(n, i-1, i-1)$ and $w(n, i-2, i-1)$.

Compute $w(n, i, i)$ and $w(n, i, i+1)$ from the forward discount factors of the cash and spot.

HEDGING

Knowledge of the whole process allows for the pricing of path-dependent options (by Monte-Carlo methods) and American options (by dynamic programming). It also allows for hedging through an equivalent spot position because the sensitivity of the options with respect to the spot can be computed. Knowing the full process, it is possible to shift the initial value and to infer the process that starts from this new value and the new price it incurs. Delta hedging can then be achieved, which will be effective throughout the life of the option if the spot behaves according to the inferred process.

It probably will not, which leads us to a more sophisticated method of hedging. We can build a robust hedge that will be efficient even if the spot does not behave according to the instantaneous inferred volatilities of the diffusion process. The idea is to associate with every contingent claim X a portfolio of European options (which should be rebalanced periodically) that will be tangential to it in the sense that it will change in value identically up to the first order for changes in the volatility manifold $\sigma(K, T)_{K,T}$.

We proceed as follows. A local move of the volatility manifold around (K_0, T_0) will lead to a new diffusion process, hence to a new value of X. We can then compute the sensitivity of X to a change of volatility $\sigma(K_0, T_0)$ and the equivalent (K_0, T_0) call position. Repeating for all (K, T), we obtain a spectrum of sensitivities Vega(K, T) and the associated (continuous) portfolio of (K, T) calls, which can be seen as a projection of X on the calls. This portfolio will behave up to the first order as X, even if the market evolves by transgressing the induced forward volatilities computed above.

CONCLUSION

Under certain conditions, it is possible to recover from the conditional laws a full diffusion process whose drift is imposed. This means that from option prices observed in the market we can induce a unique diffusion process. Clearly it would be excessive to pretend that the spot will follow this diffusion. What we can say is that the market prices European options *as if* the process was this diffusion.

In practice, this shows how a sound pricing for path-dependent and American options can be elaborated. Moreover, it finely assesses the risk of such options by performing a risk analysis along both strikes and maturities. This enables these options to be fully integrated into a book of standard European options, which is clearly a key point for many financial institutions.

1 For an account on completeness for stochastic volatility, see Dupire (1992, 1993a).

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